

# Fayet-Iliopoulos Potentials from Four-Folds

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## Abstract

We show how certain non-perturbative superpotentials  $\widetilde{W}(\Sigma)$ , which are the two-dimensional analogs of the Seiberg-Witten prepotential in 4d, can be computed via geometric engineering from 4-folds. We analyze an explicit example for which the relevant compact geometry of the 4-fold is given by  $\mathbb{P}^1$  fibered over  $\mathbb{P}^2$ . In the field theory limit, this gives an effective  $U(1)$  gauge theory with  $N = (2, 2)$  supersymmetry in two dimensions. We find that the analog of the SW curve is a  $K3$  surface, and that the complex FI coupling is given by the modular parameter of this surface. The FI potential itself coincides with the middle period of a meromorphic differential. However, it only shows up in the effective action if a certain 4-flux is switched on, and then supersymmetry appears to be non-perturbatively broken. This can be avoided by tuning the bare FI coupling by hand, in which case the supersymmetric minimum naturally corresponds to a singular  $K3$ .

## 1. Introduction

String duality has proven to be an extremely useful tool for investigating non-perturbative properties of supersymmetric gauge and other types of theories. Currently there are two complementary approaches: “geometric engineering” [1,2,3,4,5,6] makes use of the local singular geometry of compactification manifolds, while the other approach, originating in [7], uses (essentially) parallel flat  $D$ -branes to model the relevant geometry. The relation between these approaches has recently been illuminated in [8] for  $N = 2$  gauge theories in four dimensions.

The power of the geometric approach for solving gauge theories is that all gauge groups can be treated systematically in the same way. So far, most of the more concrete results that have been obtained from geometric engineering [9,1,4,6] concern  $N = 2$  supersymmetric gauge theories in  $d = 4$  [10]<sup>†</sup>, but obviously theories with  $N = 1$  supersymmetry are phenomenologically more important. Such theories have been investigated in [5], and from the  $D$ -brane point of view, for example in refs. [13,14].

It would certainly be interesting to apply similar methods to obtain non-trivial information about  $N = 1$  supersymmetric theories in four dimensions. Since such theories can be obtained from  $F$ -theory compactifications on elliptic 4-folds  $X$  [15], this suggests to study “local” (or “rigid”) mirror symmetry<sup>◇</sup> of the relevant singular geometries of these 4-folds. However,  $F$ -theory is at present still quite hard to deal with directly, and it seems a bit simpler to study instead compactifications of type IIA strings on the same kind of 4-folds. Such compactifications lead to  $N = (2, 2)$  supersymmetric theories<sup>‡</sup> in  $d = 2$ , which may be viewed as reductions of the corresponding  $N = 1$  theories in four dimensions. In type II string theory we can then use mirror symmetry in a more straightforward fashion, but nevertheless may expect that some of the relevant novel features of 4-folds can be captured in this simplified two-dimensional setting. It is the purpose of the present paper to gain some insight in how 4-folds work, by investigating a specific example.

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<sup>†</sup> For reviews see [11,12].

<sup>◇</sup> General aspects of mirror symmetry of d-folds have been first discussed in [16]; specifically 4-folds were analyzed in detail in [17] and subsequently in [18].

<sup>‡</sup> In in D-brane language, such theories have recently been investigated in [19].

## 2. Holomorphic Fayet-Iliopoulos potentials

Among of the most basic problems are 4-fold geometries that lead to gauge theories in two dimensions. Certainly gauge fields do not propagate in  $d = 2$ , but the scalar components  $\sigma$  of the  $N = (2, 2)$  supermultiplets do. More specifically, the relevant supermultiplets are the “twisted chiral” field strength multiplets,  $\Sigma \sim \sigma + \dots + \theta^+ \bar{\theta}^- (D - iF)$ , which obey  $\bar{D}_+ \Sigma = D_- \Sigma = 0$ , while matter fields correspond to the ordinary chiral multiplets  $\Phi$  with  $\bar{D}_\pm \Phi = 0$ , etc. The most general lagrangian involving these two sorts of fields [20] consists of a generalized Kähler potential  $K(\Sigma, \bar{\Sigma}, \Phi, \bar{\Phi})$  plus holomorphic chiral and twisted chiral potentials,  $W(\Phi)$  and  $\widetilde{W}(\Sigma)$ . In absence of chiral matter multiplets  $\Phi$ , the twisted chiral multiplets  $\Sigma$  are equivalent to the  $\Phi$ , and can be transformed into them [20].

Thus, integrating out massive chiral matter fields, the effective action of a gauge theory will simply be a twisted  $(2, 2)$  supersymmetric sigma-model, given by some  $K(\Sigma, \bar{\Sigma})$  plus possibly some twisted chiral potential  $\widetilde{W}(\Sigma)$ . We take the scaling dimension of  $\Sigma$  to be equal to one, so that the Kähler potential has to be multiplied by the squared inverse of a dimensionful gauge coupling and so becomes an irrelevant operator in the infrared. The other piece of the lagrangian, the twisted chiral potential, plays the rôle of a generalized Fayet-Iliopoulos term [21,22,23,19]:

$$\frac{i}{2\sqrt{2}} \int d\theta^+ d\bar{\theta}^- \widetilde{W}(\Sigma) + \text{c.c.} = -\xi(\sigma) D + \frac{\theta(\sigma)}{2\pi} F. \quad (2.1)$$

It gives rise to an effective, field dependent complex FI coupling:

$$\tau(\sigma) \equiv i\xi(\sigma) + \frac{\theta(\sigma)}{2\pi} = \widetilde{W}'(\sigma). \quad (2.2)$$

The twisted chiral potential  $\widetilde{W}(\sigma)$  is the semi-topological, holomorphic quantity that is the analog of the SW prepotential  $\mathcal{F}(a)$  [10] in four dimensions, and which is of our main interest. The FI coupling  $\tau$  is the analog of the running gauge coupling in 4d, being dimensionless and subject to RG flow. Indeed, it is known [21] that  $\tau$  receives logarithmic perturbative corrections to exactly one loop order,

$$\tau(\sigma) = \tau_0 - \frac{N}{2\pi i} \log(\sigma/\mu) + \dots, \quad (2.3)$$

where  $\mu$  is the RG scale. If we interpret the sigma-model in terms of a gauge theory, then  $N = TrQ$ , where  $Q$  is the  $U(1)$  charge of the charged chiral matter fields.

Clearly, logarithmic monodromy shifts induce shifts of the theta-angle, exactly like for  $N = 2$  gauge theories in four dimensions. For positive  $N$  and  $\text{Im}\tau$ , the theory is asymptotically free in the FI coupling, which means weakly coupled for large  $\sigma$ . We generically expect additional non-perturbative corrections to  $\tau(\sigma)$  in (2.3), the  $n$ -th instanton sector being weighted by  $\beta^n$ , where  $\beta = e^{2\pi i\tau_0}\mu^N \equiv e^{i\theta_0 - 2\pi\xi_0}\mu^N$ .

An important difference as compared to the four dimensional gauge theory is that there is a non-trivial scalar potential:

$$V(\sigma) = \frac{1}{2}|\tau(\sigma)|^2. \quad (2.4)$$

This means in particular that the vacuum energy depends on the theta-angle [24,21]. It also means that supersymmetry is broken if  $\tau(\sigma)$  is everywhere non-vanishing. Semi-classically, where we only consider the perturbative correction in (2.3), there will be  $N$  vacuum states, given by  $\sigma = \beta^{1/N}$  plus rotations by the  $\mathbb{Z}_{2N}$   $R$ -symmetry; each VEV breaks the  $R$ -symmetry to  $\mathbb{Z}_2$ .

### 3. Mirror Symmetry

Like in four dimensions, the gauge multiplets  $\Sigma$  are one-to-one to the Kähler classes belonging to  $H^{1,1}(X)$ , while the chiral matter fields correspond to the complex structure moduli belonging to  $H^{3,1}(X)$ . Since the type IIA dilaton is in a gravitational multiplet, which is real rather than twisted chiral, the holomorphic twisted chiral potential  $\widetilde{W}(\Sigma)$  does not get any type IIA space-time corrections. On the other hand, there will in general be corrections from world-sheet instantons to the Kähler sector, reflecting perturbative and non-perturbative corrections in the dual heterotic string language. The issue is to compute these corrections to  $\widetilde{W}(\Sigma)$  via mirror symmetry, which maps the type IIA string on the 4-fold  $X$  back to the type IIA string on the mirror 4-fold,  $\widehat{X}$ . In addition, the rôles of complex structure and Kähler moduli get exchanged. Thus in the mirror theory the complex structure sector is not corrected at all, as there are no 3-branes in the type IIA string that could wrap the middle homology 4-cycles, and a tree-level computation is exact.

In order to see what precise tree-level correlator we will have to compute to obtain  $\widetilde{W}(\Sigma)$ , consider the following tree-level Chern-Simons term in the  $d = 10$  type IIA string:

$$\mathcal{L}_{CS} = B \wedge F_4 \wedge F_4, \quad (3.1)$$

where  $F_4$  is the field strength of the 3-form field. On the 4-fold  $X$  we can then expand

$$B = \sigma \mathcal{O}^{(1)} , \quad F_4 = \nu \mathcal{O}^{(2)} + F \wedge \mathcal{O}^{(1)} , \quad (3.2)$$

where  $\mathcal{O}^{(i)}$  represent elements of  $H_{\partial}^{i,i}(X, \mathbb{Z})$ . The occurrence of 4-forms  $\mathcal{O}^{(2)}$  [17,18] and the related scalars  $\nu$  is a novel feature as compared to usual 3-fold compactifications. The important point is that at the quantum level,  $\nu$  is an integer c-number:

$$\int_{\mathcal{C}^4} F_4 = \nu \in \mathbb{Z} \quad (3.3)$$

(or possibly  $\nu \in \frac{1}{2}\mathbb{Z}$ ), which is known as “4-flux” [25,26,27]. Therefore (3.1) leads to a two-dimensional FI coupling of the form  $\sigma F \nu \langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(2)} \rangle$ , which means that

$$\widetilde{W}''(\sigma) = \nu \langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(2)} \rangle_{IIA} . \quad (3.4)$$

We see that the FI coupling is proportional to the 4-flux, and since non-trivial FI terms typically do exist,  $\nu$  will generically have to be non-zero. On the other hand, unbroken supersymmetry tends to favor  $\nu = 0$ , though this is not strictly required [26,28]. We will come back to this point later in section 5.2.

One might wonder about fivebrane instantons wrapping around appropriate 6-cycle divisors, producing corrections to the Kähler sector that may not be captured by mirror symmetry; indeed such instantons do lead to potentials in  $M$ - and  $F$ -theory compactifications to three and four dimensions [29,30]. There is a simple argument why such instantons do not contribute to  $\widetilde{W}$ , the reason being the non-zero flux  $\nu$ . More specifically, it is known that on the fivebrane world volume, there is a self-dual 2-form field with field strength  $T_3$  which obeys  $dT_3 = F_4$ . As pointed out in [31], from this follows that a wrapped 5-brane implies  $F_4$  to be cohomologically trivial, ie.,  $\nu = 0$  in (3.3). Conversely, a non-zero 4-flux (emanating from a submanifold  $\mathcal{C}^4$  of the 6-cycle) prohibits the wrapping of the fivebrane, and thus there are no fivebrane instanton corrections to  $\widetilde{W}$ .<sup>†</sup>

The 3-point function in (3.4) is a correlator in topological field theory that can be evaluated via mirror symmetry [16,17,18]. When twisting by the internal  $U(1)$  current we can project on either the chiral or on the twisted chiral subsector of the

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<sup>†</sup> This makes sense also from the point of view of  $M$ - and  $F$ -theory compactifications on  $X$ , where  $\nu \langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(2)} \rangle_{\text{class}}$  is the coefficient of Chern-Simons and GS anomaly cancelling terms, respectively. Such topological couplings are supposed not to be corrected.

theory. When twisting left-right symmetrically, the background charges are  $(-4, -4)$  and thus we project on the  $\Sigma$ -sector where the basic correlators are

$$C_{112} = \langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(2)} \rangle, \quad C_{1111} = \langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(1)} \rangle. \quad (3.5)$$

Both  $\mathcal{O}^{(1)}$  and  $\mathcal{O}^{(2)}$  survive the topological twist (as they obey  $h = q/2$ ), even though the 4-form operators do not represent continuous but discrete moduli of the TFT. Note that only the three-point function and not the four-point function contributes to a holomorphic potential. Indeed after twisting (and setting  $\mathcal{O}^{(i)} \rightarrow \mathcal{O}^{(i)} e^{-\phi - \bar{\phi}}$ ), one needs exactly three and not four operators in the  $(-1, -1)$ -picture, and therefore one has to insert an extra picture changing operator in the four-point correlator. This introduces momentum factors which means that the four-point function contributes to the Kähler potential and not to  $\widetilde{W}$ .

Note also that on 3-folds, the 4-forms are dual to 2-forms so that they are not independent. In contrast, for 4-folds the  $H^{2,2}$  sector is independent and generically quite large, and in general only a small subset of  $H^{2,2}$  will be related to the sub-sector we are interested in (the "primary subspace" generated by wedging the  $(1,1)$ -forms). In particular, for theories with one modulus, the relevant 4-form is simply  $\mathcal{O}^{(2)} \sim (\mathcal{O}^{(1)})^2$ , and therefore we have from factorization [16]:

$$C_{1111} \sim (C_{112})^2. \quad (3.6)$$

#### 4. An example: $\mathbb{P}^2$ fibered over $\mathbb{P}^1$

The best understood example for geometric engineering is pure  $N = 2$  Yang-Mills theory [10] in  $d = 4$ , where the relevant type IIA 3-fold geometry is given by an  $A_1$  singularity fibered over  $\mathbb{P}^1$  [1]. Since such a singularity describes a vanishing 2-sphere, the local geometry is effectively given by a fibration of  $\mathbb{P}^1$  over  $\mathbb{P}^1$ , ie., by a Hirzebruch surface.<sup>◇</sup> The type IIB mirror geometry of this fibration is indeed exactly given by (a non-compact form of) the SW curve [4,6].

Our intention is to stay as close as possible to this situation, and simply to try to see what will come out for a 4-fold as compared to a 3-fold. The closest relative of

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◇ Together with the normal bundle on it, this yields a non-compact 3-fold whose compact part is the fibration. We will in the following not explicitly mention the non-compact parts of 3- or 4-folds.

the SW geometry for a 4-fold is given by a fibration of the same  $\mathbb{P}^1$  over a  $\mathbb{P}^2$  base. In the field theory limit, one naively expects this to give rise to a reduction on  $\mathbb{P}^2$  of a six-dimensional  $SU(2)$  gauge theory down to two dimensions, the  $SU(2)$  arising from the fiber  $\mathbb{P}^1$ . At any rate, whether this expectation bears out or not, the low-energy effective theory that we will obtain is a  $U(1)$  gauge theory, and our task is to compute its twisted chiral potential  $\widetilde{W}$ .

On general grounds [17,18], the contributions from world-sheet instantons (including multi-covers) to the potential will be of the form  $\widetilde{W} = Q(t_f, t_b) + \sum n_{i,j} \text{Li}_2(q_f^i q_b^j)$  (where  $Q$  is a quadratic function of the Kähler moduli associated with fiber and base, and  $q_{f,b} \equiv e^{2\pi i t_{f,b}}$ ). It will turn out, exactly like in four dimensions [4], that wrappings of world-sheet instantons around the  $\mathbb{P}^1$  fiber produce, in the rigid limit, the logarithmic one-loop contribution to  $\widetilde{W}$  (from  $\text{Li}_2(1 + \sqrt{\alpha'}\sigma) \sim \text{const} + \sqrt{\alpha'}\sigma \log \sigma$ ), and wrappings around the various classes of the  $\mathbb{P}^2$  base give additional non-perturbative corrections.

#### 4.1. Picard-Fuchs system

Taking toric geometry as starting point, we choose the following Mori (charge) vectors to describe the fibration data of the non-compact 4-fold:

$$\begin{aligned} l_f &= (-2, 1, 0, 0, 0, 1), & (\mathbb{P}^1 \text{ fiber}), \\ l_b &= (0, 0, 1, 1, 1, -3), & (\mathbb{P}^2 \text{ base}). \end{aligned} \tag{4.1}$$

Following standard methods (see eg., [12]), the non-compact mirror then looks, up to quadratic pieces:

$$W = z_b x_2^4 + z_f x_1^2 x_3 x_4 + x_1 x_2 x_3 x_4 + x_2^2 x_3 x_4 + x_2 x_3^2 x_4 + x_2 x_3 x_4^2 \tag{4.2}$$

(which involves the canonical parameters:  $z_f = \frac{a_2 a_6}{a_1^2}$ ,  $z_b = \frac{a_3 a_4 a_5}{a_6^3}$ ). The associated Picard-Fuchs system is ( $\theta_f \equiv z_f \partial_{z_f}$ , etc):

$$\begin{aligned} L_f &= \theta_f(\theta_f - 3\theta_b) - 2z_f \theta_f(2\theta_f + 1) \\ L_b &= \theta_b^3 - z_b(\theta_f - 3\theta_b - 2)(\theta_f - 3\theta_b - 1)(\theta_f - 3\theta_b), \end{aligned}$$

and we find for the relevant components of the discriminant:

$$\begin{aligned} \Delta_1 &= (1 + 27z_b)^4 \\ \Delta_2 &= (1 - 4z_f)^3 - 1728z_f^3 z_b. \end{aligned}$$

Note the cubic splitting of the classical  $A_1$  singularity at  $z_f = 1/4$ . The rigid field theory limit we are interested in, amounts to taking the base  $\mathbb{P}^2$  large and the fiber  $\mathbb{P}^1$  small. This is localized in the moduli space at the point of tangency  $z_f = 1/4$ ,  $z_b = 0$ , which needs to be properly blown-up [9]. Suitable variables for this double-scaling limit are given by:

$$\begin{aligned} z_1 &= 4z_f - 1 \equiv \alpha' u \\ z_2 &= -\frac{3z_b^{1/3}}{4z_f - 1} \equiv \frac{\beta^{1/3}}{u} . \end{aligned}$$

which leads to  $\Delta_2 \sim \alpha'^3(u^3 - \beta) + O(\alpha'^4)$ . Hence we are left with only one independent variable in the rigid limit  $\alpha' \rightarrow 0$ , so that effectively one  $U(1)$  factor decouples. We also see that by dimensional transmutation a scale  $\mu$  is introduced in this process,  $z_b \sim e^{-S} \sim (\alpha')^3 \mu^6 e^{2\pi i \tau_0}$ , where  $S$  is the heterotic dilaton. This means that  $\beta \equiv e^{2\pi i \tau_0} \mu^6$ , and that  $u$  has mass dimension 2. From now on, we will mostly set  $\beta = 1$ .

After rescaling the periods, the above PF system reduces to the following differential operator:

$$\begin{aligned} L_f &\rightarrow 0 \\ L_b &\rightarrow L_R \equiv (2\theta_2 + 1)^3 - 8z_2^3 \theta_2 (\theta_2 + 1) (\theta_2 + 2) . \end{aligned}$$

Note that this operator, coming from the base  $\mathbb{P}^2$  and not from the fiber, is of third order, which is the natural order of a PF operator associated with a 2-fold. In terms of the inverse variable  $u = 1/z_2$ , it constitutes a generalized hypergeometric equation of type<sup>†</sup>  ${}_3F_2(-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}; \frac{1}{3}, \frac{2}{3}; u^3)$ , with standard series solutions around  $u = 0$ .

The moduli space thus has three singularities at  $u^3 = 1$ , plus one at  $u = \infty$ , which is the weak-coupling limit of the rigid theory. In the weak-coupling region, the three solutions behave as  $\sigma \sim \sqrt{u}$ ,  $\tilde{\sigma}_{D1} \sim \sqrt{u} \log u$ ,  $\tilde{\sigma}_{D2} \sim \sqrt{u} (\log u)^2$ , where  $\sigma$  represents the scalar component of the  $U(1)$  gauge superfield  $\Sigma$ . Since  $u$  has mass dimension two,  $\sigma$  has mass dimension equal to one, and this exactly what we had assumed in section 2.

Moreover, we find that the derivatives

$$\frac{\partial}{\partial u} \begin{pmatrix} \sigma \\ \sigma_{D1} \\ \sigma_{D2} \end{pmatrix} (u) \equiv \begin{pmatrix} \omega \\ \omega_{D1} \\ \omega_{D2} \end{pmatrix} (u) \quad (4.3)$$

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<sup>†</sup> For  $SU(2)$  SW theory in  $d=4$ , the analogous system is of type  ${}_2F_1(-\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; u^2)$ .



are solutions of a generalized hypergeometric equation of type  ${}_3F_2(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}; \frac{1}{3}, \frac{2}{3}; u^3)$ . These correspond to the periods of the holomorphic 2-form,  $\Omega_{2,0} \equiv (\frac{\partial W_{K3}}{\partial x})^{-1} \frac{dz}{z} \wedge \frac{dw}{w}$ , that is associated with the following “rigid”  $K3$  surface:

$$W_{K3} = z + w - \frac{1}{27zw} + (x^2 + u) = 0. \quad (4.4)$$

This  $K3$ , which arises from (4.2) in the rigid limit after appropriate rescalings, is the 2d analog of the elliptic curve [10] in four dimensions. Indeed from (4.3) we conclude, in complete analogy to SW theory, that  $\sigma, \sigma_{D1}, \sigma_{D2}$  are periods of a specific meromorphic 2-form

$$\lambda_2 = -2x \frac{dz}{z} \wedge \frac{dw}{w} = -2i \sqrt{z + w - \frac{1}{27zw} + u} \frac{dz}{z} \wedge \frac{dw}{w}$$

on the auxiliary  $K3$  surface, which has the characteristic property

$$\frac{\partial}{\partial u} \lambda_2 = \Omega_{2,0}. \quad (4.5)$$

#### 4.2. Properties of the periods

Note that  $K3$  periods are algebraically dependent,  $\omega \omega_{D2} \sim \omega_{D1}^2$ , whence there is only one independent ratio:

$$\tau_{K3}(u) = \frac{\partial_u \sigma_{D1}(u)}{\partial_u \sigma(u)}, \quad \frac{\partial_u \sigma_{D2}(u)}{\partial_u \sigma(u)} = (\tau_{K3}(u))^2. \quad (4.6)$$

In fact, it is known for some while [32,33] that the PF equation of a single-modulus  $K3$  can be reduced to a Schwarzian differential equation for  $\tau$ . Specifically, following [32] we find from the  $K3$  PF equation:

$$\{\tau_{K3}(u); u^3\} = \frac{36u^6 - 41u^3 + 32}{72u^6(u^3 - 1)^2}, \quad (4.7)$$

which implies that

$$u^3(\tau_{K3}) = \frac{1}{1728} j(\tau_{K3}), \quad (4.8)$$

where  $j$  is the well-known modular invariant. Hence the monodromy group generated in the  $u^3$ -plane (with singularities at  $0, 1, \infty$ ) acting on  $\tau_{K3}$  is the modular group  $SL(2, \mathbb{Z})$ ; this is indeed a typical feature of  $K3$  surfaces [33]. The monodromy group for our preferred parametrization, which is the  $u$ -plane, is then the corresponding

index 3 normal subgroup  $\Gamma_1$  [34] with branch scheme  $(1, 3, 3)$  (with  $T^3 : \tau_{K3} \rightarrow \tau_{K3} + 3$  as one of its generators).

The specific linear combinations of the PF solutions that correspond to the integral geometric periods can be determined in various ways, eg., by considering the asymptotic expansion of the period integrals. The geometric periods we find turn out to be most simply expressed in terms of their derivatives, ie., in terms of the standard  $K3$  periods (4.3). This is because the  $K3$  periods can be written [32,33] in terms of ordinary hypergeometric functions, and this is very convenient for analytic continuation. Explicitly, we find for the geometric periods  $\varpi \equiv (\sigma, \sigma_{D1}, \sigma_{D2})^t$  (up to integral changes of basis):

$$\frac{\partial}{\partial u} \varpi \equiv \begin{pmatrix} \omega \\ \omega_{D1} \\ \omega_{D2} \end{pmatrix} = \begin{pmatrix} \zeta_0^2 \\ \zeta_0 \zeta_1 - i \zeta_0^2 \\ \zeta_1^2 - 2i \zeta_0 \zeta_1 - \zeta_0^2 \end{pmatrix}, \quad (4.9)$$

where

$$\begin{aligned} \zeta_0 &= (u^3 - 1)^{1/12} {}_2F_1\left(\frac{1}{12}, \frac{7}{12}, 1, \frac{1}{1 - u^3}\right) \\ \zeta_1 &= i\gamma {}_2F_1\left(\frac{1}{12}, \frac{1}{12}, \frac{1}{2}, 1 - u^3\right), \quad \text{with } \gamma \equiv \frac{\Gamma(1/12)\Gamma(5/12)}{2\pi^{3/2}}. \end{aligned}$$

At weak coupling, the precise forms of the leading terms are:

$$\begin{aligned} \sigma(u) &= 2\sqrt{u} + O(u^{-5/2}) \\ \sigma_{D1}(u) &= \frac{3i}{\pi} \sqrt{u} (\log[12u] - 2) + O(u^{-5/2}) \\ \sigma_{D2}(u) &= -\frac{9}{2\pi^2} \sqrt{u} (\log[12u]^2 - 4\log[12u] + 8) + O(u^{-5/2}). \end{aligned} \quad (4.10)$$

The periods turn out to obey the following identity:

$$u(\sigma) = -\frac{72}{\pi^2} \varpi \cdot C \cdot \varpi, \quad \text{with intersection form } C \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (4.11)$$

which reflects the algebraic dependence of the  $K3$  periods. Using the well-known formulas for the analytic continuation of hypergeometric functions, and (4.11) for

fixing the integration constants, we find for the periods near the singularity  $w \equiv (u - 1) \rightarrow 0$ :

$$\begin{aligned}\sigma(w) &= \left(\frac{\gamma^2}{4} + \frac{36}{\pi^2 \gamma^2}\right) + \frac{1}{4}w\gamma^2 - \frac{2w^{3/2}}{\sqrt{3}\pi} + O(w^2) \\ \sigma_{D1}(w) &= \left(\frac{i\gamma^2}{4} - \frac{36i}{\pi^2 \gamma^2}\right) + \frac{1}{4}iw\gamma^2 + O(w^2) \\ \sigma_{D2}(w) &= -\left(\frac{\gamma^2}{4} + \frac{36}{\pi^2 \gamma^2}\right) - \frac{1}{4}w\gamma^2 - \frac{2w^{3/2}}{\sqrt{3}\pi} + O(w^2)\end{aligned}\tag{4.12}$$

Note their power-like, non-logarithmic behavior near the strong coupling singularity. From (4.10) and (4.12) we infer the following monodromy matrices (acting on  $\varpi$ ) associated with  $u = \infty, 1$ :

$$\begin{aligned}M_\infty &= \begin{pmatrix} -1 & 0 & 0 \\ -3 & -1 & 0 \\ -9 & -6 & -1 \end{pmatrix} \equiv T^3 \\ M_1 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \equiv S, \quad \text{with } S^2 = \mathbb{1},\end{aligned}\tag{4.13}$$

and similar (via conjugation) for the other two strong coupling singularities. We observe an ‘‘S-duality’’ at  $u = 1$ , associated with  $\tau_{K3}(0) \equiv \frac{\partial_w \sigma_{D1}}{\partial_w \sigma}|_{w=0} = i$ . Note also that in contrast to an elliptic curve, where the point of enhanced  $\mathbb{Z}_2$  symmetry corresponds to a smooth curve, this point corresponds here to a singular  $K3$  with vanishing period  $\sigma + \sigma_{D2}|_{(w=0)} = 0$ .

#### 4.3. The quantum FI coupling

We are now equipped to compute correlation functions. The 4-point function is particularly easy to compute, because it is encoded in the PF system. One can actually obtain the rigid coupling directly from the rigid PF equation, which is even simpler. Concretely, the classical coupling is defined by

$$C_{uuuu} = - \int_X \Omega_{4,0} \wedge \partial_u^4 \Omega_{4,0}.$$

In the rigid limit, and integrating out two dimensions exactly as in [1], the holomorphic 4-form turns into the meromorphic 2-form on the  $K3$ , so that

$$C_{uuuu} = \int_{K3} (\partial_u \lambda_2) \wedge \partial_u^3 \lambda_2 = \int_{K3} \Omega_{2,0} \wedge \partial_u^2 \Omega_{2,0}.$$

where we used (4.5). We can then use that the periods (4.3) satisfy the hypergeometric system of type  ${}_3F_2(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}; \frac{1}{3}, \frac{2}{3}; u^3)$ , which is of the form  $L_{K3} = \sum f_k(u) \partial_u^k$ . Following a similar strategy as in [35], by defining

$$V^{(k)}(u) \equiv \int_{K3} \Omega_{2,0} \wedge (\partial_u)^k \Omega_{2,0} = \omega(\partial_u)^k \omega_{D2} + \omega_{D2}(\partial_u)^k \omega + \kappa \omega_{D1}(\partial_u)^k \omega_{D1}$$

(where the self-intersection number  $\kappa$  is any constant), we know that

$$0 = \sum f_k(u) V^{(k)}(u) \equiv (1 - u^3) V^{(3)} - \frac{9}{2} V^{(2)} u^2 - \frac{13}{4} V^{(1)} u - \frac{1}{8} V^{(0)} .$$

On the other hand,  $V^{(0)} = V^{(1)} = 0$  (due to the algebraic dependence of the periods) and  $V^{(3)} = \frac{3}{2} \partial_u V^{(2)} - \frac{1}{2} \partial_u^2 V^{(1)}$ , whence  $3u^2 V^{(2)} + (u^3 - 1)(V^{(2)})' = 0$ . This has as solution

$$V^{(2)}(u) \equiv C_{uuuu} = \frac{1}{u^3 - 1} ,$$

something one might have guessed beforehand. Using  $u \sim \sigma^2 + O(\sigma^{-4})$  the coupling in the flat variable  $\sigma$  thus is

$$C_{\sigma\sigma\sigma\sigma} = \frac{1}{u(\sigma)^3 - 1} \left( \frac{\partial u(\sigma)}{\partial \sigma} \right)^4 \sim \frac{1}{\sigma^2} + O(\sigma^{-8}) .$$

We are however interested in the 3-point coupling  $C_{\sigma\sigma} \equiv C_{112}$ ; from factorization of amplitudes (3.6) we have that  $C_{\sigma\sigma\sigma\sigma}(\sigma) \sim (C_{\sigma\sigma}(\sigma))^2$  which gives  $C_{\sigma\sigma} \sim 1/\sigma + \dots$ . By (3.4), this can be two times integrated to finally give the twisted chiral potential. In fact, we find that

$\widetilde{W}(\sigma) = \sigma_{D1}(\sigma)$

(4.14)

up to integration constants. This reflects the result [17,18] for the non-rigid case, that the cubic couplings  $C_{112}$  are given by second derivatives of the “middle” periods with respect to flat coordinates. Comparing to (4.6) it thus follows that the Fayet-Iliopoulos coupling  $\tau$  in (2.2) coincides with the modular parameter of the auxiliary  $K3$  surface, up to a possible integration constant:

$$\begin{aligned} \tau(\sigma) &\equiv \widetilde{W}'(\sigma) = \tau_{K3}(\sigma) \\ &= \tau_0 - \frac{6}{2\pi i} \left[ \log\left(\frac{\sigma}{\mu}\right) - \frac{25}{432} \frac{\beta}{\sigma^6} - \frac{9449}{497664} \left(\frac{\beta}{\sigma^6}\right)^2 - \frac{4521065}{483729408} \left(\frac{\beta}{\sigma^6}\right)^3 + \dots \right]. \end{aligned} \tag{4.15}$$

where the bare coupling is  $\tau_0 = \frac{3i}{2\pi} \log 3$ . Above, we inferred the normalization, ie. the factor  $N = 6$ , from  $u^3 = j(\tau)$ , and we have reinstated the dependence on  $\beta \equiv e^{2\pi i \tau_0} \mu^6$ . Because  $N$  is positive, the theory is indeed asymptotically free.

Note that  $\widetilde{W}(\sigma)$ , being given by a period integral,<sup>†</sup> transforms non-trivially under monodromies induced by looping around the singularities in the moduli space. It is thus a section and not a function, as it is usual for holomorphic quantities in supersymmetric theories. The identification  $\widetilde{W}(\sigma) = \sigma_{D1}$  also makes sense from the point of view of central charges: in analogy to  $N = 2$  gauge theory in 4d, one would be tempted to write for the central charge of the superalgebra:  $Z = n\sigma + m\sigma_{D1} + k\sigma_{D2}$ , and it is indeed well-known [36] that in 2d the superpotential figures in the central charge.

Formally,  $n, m, k$  correspond to quantum numbers associated with  $D2, D4, D6$  branes of the type IIA string wrapped around 2, 4, 6-cycles of the 4-fold, respectively. In this sense the singularity at  $u = 1$  would be associated with a massless state with  $(n, m, k) = (1, 0, 1)$ . However, such an interpretation appears to be problematical in  $d \leq 3$ , because of divergences the BPS mass formula does not make much sense for fields charged under local gauge currents [37].

#### 4.4. Rigid Special Geometry

One might a priori expect some generalization of rigid special geometry [38] to constrain the Kähler potential  $K(\Sigma, \overline{\Sigma})$ .<sup>◇</sup> Indeed there is a natural expression for a Kähler potential for any  $d$ -fold, given by  $e^{-K} = \int \Omega \wedge \overline{\Omega}$ . In the rigid limit, the natural expression for our rigid Kähler potential thus is

$$\begin{aligned} K(\sigma, \overline{\sigma}) &= \varpi \cdot C \cdot \overline{\varpi} = \sigma \overline{\sigma}_{D2} + \overline{\sigma} \sigma_{D2} - 2\sigma_{D1} \overline{\sigma}_{D1} \\ &\sim \sigma \overline{\sigma} + \sigma \overline{\sigma} (\log(\sigma) + \log(\overline{\sigma}))^2 + \dots, \end{aligned} \tag{4.16}$$

which leads to the following metric:

$$g(\sigma, \overline{\sigma}) \equiv \partial \overline{\partial} K(\sigma, \overline{\sigma}) = (\text{Im } \tau(\sigma))^2, \tag{4.17}$$

where  $\tau$  is the FI coupling as above. Note that the metric (4.17) is invariant under discrete theta-shifts,  $\tau \rightarrow \tau + m$ , despite of the unusual asymptotic form of the Kähler

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<sup>†</sup> Superpotentials identified with period integrals recently came up in  $N = 1$  SQCD [14].

<sup>◇</sup> Of course, the Kähler potential is not protected from quantum corrections, so this is to be taken *cum grano salis*.

potential (4.16). Indeed  $K$  looks similar but different as compared to the familiar one-loop Kähler potentials of the  $CP^1$  model or the  $SU(2)$  gauge theory. The latter has the form [39]  $K_{1-loop} = \int_{e^2} \frac{dy}{y} \log(1 + \frac{\sigma\bar{\sigma}}{y}) = \text{Li}_2(\frac{\sigma\bar{\sigma}}{e^2}) \sim \log(\sigma)\log(\bar{\sigma})$ , which lacks the prefactor  $\sigma\bar{\sigma}$  of our Kähler potential (as it must be on dimensional grounds; in contrast, our field  $\sigma$  has scaling dimension equal to one). At any rate, the important structure is the presence of quadratic logarithms, and this is a characteristic property of “rigid” 2-folds.

The generalization of the well-known special geometry relation,  $R \sim 2g^2 + e^K g^{-1} C_{111} \overline{C_{111}}$  [40], for 3-folds to 4-folds has been discussed in [16]. In the rigid limit, this generalization takes the form  $R_{\sigma\bar{\sigma}\sigma\bar{\sigma}} \sim (R^{-1})^{\sigma\bar{\sigma}\sigma\bar{\sigma}} C_{\sigma\sigma\sigma\sigma} \overline{C_{\sigma\sigma\sigma\sigma}}$ , which in view of the factorization property (3.6) is equivalent to

$$R_{\sigma\bar{\sigma}\sigma\bar{\sigma}} = 2C_{\sigma\sigma} \overline{C_{\sigma\sigma}}. \quad (4.18)$$

Using the couplings  $C_{\sigma\sigma} = \tau'$  as well as  $R = g\bar{\partial}g^{-1}\partial g$ , it is trivial to check that this relation is indeed satisfied, showing consistency of our results. Note also that by taking derivatives of the metric (4.17) and using (3.6), we get among other terms the following term:  $\sigma\sigma\partial\sigma\bar{\partial}\bar{\sigma}C_{\sigma\sigma\sigma\sigma}$ . This shows that the holomorphic four-point correlator (3.5) indeed contributes to the non-holomorphic Kähler potential.

## 5. Discussion

### 5.1. 4-Flux and matter

In our treatment of the example we have so far been tacitly neglecting the fact that the FI potential  $\widetilde{W}$  really is multiplied by an integer number, the 4-flux  $\nu$  in (3.4). The mirror symmetry computation we have done concerns only the correlator  $C_{112} = \langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(2)} \rangle$ , and is insensitive to the overall factor  $\nu$ .

The superpotential  $\widetilde{W}$  (4.14) that we have been computing thus only, but then necessarily appears if we turn on the 4-flux (if we were free to do so, see below). This has various implications: first, since  $TrQ$  is proportional to  $\nu$ , this is, from a field theory point of view, equivalent to switching on charged matter. That is, in the effective lagrangian we cannot distinguish the logarithmic term  $\frac{6\nu}{2\pi i} \Sigma \log \Sigma$  in  $\widetilde{W}$  that we get from geometry, from a field theory one-loop term  $\frac{TrQ}{2\pi i} \Sigma \log \Sigma$ , generated by integrating out massive chiral matter multiplets. In other words, the effective theory

behaves exactly in the way as there would be charged matter, although geometrically we did not put any in.

This may also have a bearing on chiral matter in  $N = 1$  theories in four dimensions, where there generically is a Green-Schwarz anomaly cancelling term of the form:  $\mathcal{L}_{GS} = c (B \wedge F)$  with  $c \equiv \text{Tr} Q$ . In F-theory compactified on a 4-fold, this term is obtained from the following formal coupling [41] in twelve dimensions:  $\mathcal{L} = A_4 \wedge F_4 \wedge F_4$ , where  $A_4$  is a 4-form gauge field. That is, upon expanding  $A_4 = B \wedge \mathcal{O}^{(1)}$  and  $F_4$  as in (3.2), one obtains this coupling with  $c = \nu \langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(2)} \rangle_{\text{class}}$ , where "class" denotes the classical intersection (what figured in  $d = 2$  was the world-sheet corrected quantum version of this). This means that whenever there is an anomalous  $U(1)$  in  $d = 4$ , in F-theory language some 4-flux  $\nu$  must to be non-zero. Turning this around, switching on  $\nu$  may in some sense *implement* chiral matter.

In addition it must be that  $SU(2)$  is broken when we switch on  $\nu$ , simply because  $\text{Tr} Q$  vanishes identically for any non-abelian group. Indeed, the resulting potential  $\int \nu \widetilde{W}(\Sigma) \sim \nu(\Sigma + \Sigma \log \Sigma + \dots)$  is not invariant under discrete Weyl transformations ( $\Sigma \rightarrow -\Sigma$ ). However only mildly so: it just changes sign, up to an additional theta-shift. Weyl invariance of the effective theory can therefore be restored if we simultaneously flip the signs of  $\sigma$  and of the symmetry breaking flux  $\nu$ . In this way the theory exhibits the presence of the  $SU(2)$  that was originally built in.

## 5.2. Supersymmetric vacua ?

Another important implication of turning on 4-flux is a non-vanishing scalar potential (2.4),  $V \sim \nu^2 |\tau(\sigma)|^2$ , where the FI coupling is given in terms of the  $K3$  periods (4.9) as  $\tau = \tau_{K3} \equiv \omega_{D1}(\sigma)/\omega(\sigma)$ . One may wonder whether there are any supersymmetric vacua given by  $\tau = 0$ .<sup>†</sup> Semiclassically, where  $\tau \sim \tau_0 - \frac{6}{2\pi i} \log \sigma$ , supersymmetric vacua obviously do exist.

But non-perturbatively,  $\tau$  is non-zero everywhere over the moduli space, because it is a modular parameter living in three copies of the usual fundamental region. This means that supersymmetry must be spontaneously broken ! One might say that this happens because the relevant 2-cycle has non-zero "quantum volume" [43] throughout

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<sup>†</sup> This ties together with the following condition for a supersymmetric 4-form background [28]:  $(F_4)_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} \tilde{J}^{\tilde{c}\tilde{d}} = 0$ , where  $J$  is the Kähler form. This Uhlenbeck-Yau type of equation implies [42]  $\int \tilde{J} \wedge J \wedge F_4 = 0$ , which leads to [17]  $\sum t_i t_j \nu_k \langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \mathcal{O}^{(2)} \rangle = 0$ , where  $t_i$  are special coordinates. We recognize here essentially the condition for a vanishing FI potential.

the moduli space. One could have speculated that there would be a zero quantum volume at the conifold point  $u = 1$ , similar to what happens for certain vanishing del Pezzo 4-cycles [44], but it did not turn out that way. Specifically we have at the conifold point  $\tau = i$ , which corresponds to a non-vanishing FI parameter,  $\xi = 1$ . Note that in view of (4.12), the physics at the singularity is governed by a power-like, and not logarithmic, potential.

The story is however not that clearcut, as  $\nu$  cannot, in general, be adjusted at will. Indeed from (3.1) one has in addition a tadpole  $B\nu^2\langle\mathcal{O}^{(2)}\mathcal{O}^{(2)}\rangle$ , besides the 1-loop tadpole  $B\int I_8(R)$  [45], and all these tadpoles need to be cancelled together with contributions from extra 1-branes [46]; this sometimes even forces  $\nu$  to be non-zero [27,47]. These considerations involve global properties of the compactification 4-fold  $X$ , while we were discussing in our example only local properties. We neglected in particular possible effects of extra tadpole cancelling branes. It may thus well be that in our local analysis, we did not capture an important “global” ingredient, like a contribution to the vacuum energy. What we have in mind is that there might be an extra bare FI coupling in the potential,

$$\widetilde{W}(\Sigma) \longrightarrow \widetilde{W}(\Sigma) + \delta \Sigma \tag{5.1}$$

which shifts the vacuum energy and restores a supersymmetric vacuum. In fact, in our computation of  $\widetilde{W}$  in (4.14) there was room for an integration constant of exactly that type, so all we can really say from our computation is that  $\tau = \tau_{K3} + \delta$ , though a non-zero  $\delta$  would not be natural from the geometrical point of view. Note that adding such a constant also does not violate the special geometry relation (4.18).

It would appear particularly appealing to add such a term with  $\delta = -i$ .<sup>◇</sup> Namely without it, the potential  $V \sim |\tau|^2$  would have a flat direction along the lower boundary of the fundamental region (the arc between  $\text{Im}\tau = -\frac{1}{2}$  and  $\text{Im}\tau = +\frac{1}{2}$ ), which would not seem to make much sense. After adding this term, the vacuum degeneracy is resolved, a mass gap created and there is a supersymmetry preserving ground state located at the self-dual conifold point  $u = 1$  ( $\tau_{K3} = i$ ). This would be in line with the arguments of [26] that say when switching on  $p$ -form fluxes, potentials are generated whose minima lie on conifold points.

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◇ For simplicity, we will set  $\nu = 1$  in the following



The other two images of the conifold point (located at  $u = (-1)^{\pm 2/3}$ ) correspond to  $\tau_{K3} = i \pm 1$ , ie., to non-vanishing theta-angles  $\pm 2\pi$ . Physically, a non-zero theta-angle describes a constant electric field that contributes to the vacuum energy [24]. The point is that  $\theta$  naturally lives in the domain  $-\pi \leq \theta \leq \pi$ , because for  $|\theta| > \pi$  the vacuum energy can be reduced by pair creation to  $|\theta| \leq \pi$ . Semi-classically, one thus defines [21] an effective theta-angle in terms of a piecewise smooth function  $\tilde{\theta} = \theta + 2\pi n$ , such that  $|\tilde{\theta}| \leq \pi$ .<sup>‡</sup>

In this sense there are then six semi-classical vacua given by  $\sigma_n = e^{2\pi i(\tau_0 + n)/6}$ ,  $n = 0, \dots, 5$ , because all  $\sigma_n$  lead to  $\tilde{\theta} = 0$ . This gives a non-zero Witten index, which precludes a spontaneous breakdown of supersymmetry. If we add  $\delta = -i$ , then we consistently have, in the same sense, also at the non-perturbative level six vacua, coming from the three singularities at  $u^3 = 1$ , each singularity counting two vacua because of the enhanced  $\mathbb{Z}_2$  symmetry there.

### 5.3. Generalizations

One can easily see that all fibrations of  $\mathbb{P}^1$  over  $\mathbb{P}^2$  give the same rigid limit. In our example, one may as well take for instance  $\mathbb{P}^1 \times \mathbb{P}^2$  with  $l_f = (1, 1, 0, 0, 0, -2)$  and  $l_b = (0, 0, 1, 1, 1, -3)$ , instead of (4.1). This is completely analogous to four dimensions, where all the fibrations of  $\mathbb{P}^1$  over  $\mathbb{P}^1$  (the Hirzebruch surfaces  $F_n$ ), give in the rigid limit the same result, ie., the  $SU(2)$  Seiberg-Witten curve, for all  $n$  [4,6]. Thus there is an analogous universality for the 2d rigid theories based on  $\mathbb{P}^2$ .

Note however that since the base is two-dimensional, there is more than only one choice for it:  $\mathbb{P}^2$  just corresponds to the simplest possibility with the lowest number of parameters. The ubiquitous factor of 3 we encounter (giving the order of splitting of the classical singularity, the order of the PF operator, the value of  $N = 6 \equiv 2 \times 3$  and the  $\mathbb{Z}_6$  symmetry of the instanton expansion) traces back to the intersection of  $c_1(\mathbb{P}^2)$  with the  $h_{1,1}$  class, which is 3. For other choices of the base there will be other such characteristic numbers, and therefore these theories will be different from the model we have been discussing.

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<sup>‡</sup> A natural way to restore the  $2\pi$ -periodicity of the effective theta-angle in the non-perturbative theory, would be simply go to a triple cover of the moduli space, parametrized by  $\tilde{u} \equiv u^3 = j(\tau_{K3})$ , yielding a smooth and  $2\pi$ -periodic function for  $\theta(\tilde{u})$ . This parametrization would be natural from the viewpoint of  $K3$  geometry, but not natural from the viewpoint of a fibered  $A_1$  singularity.

One can for instance also consider fibrations of  $\mathbb{P}^1$  over Hirzebruch surfaces  $F_n$ , where the global symmetry is  $\mathbb{Z}_8 = (\mathbb{Z}_2)^3$  instead of  $\mathbb{Z}_6$ . A new feature will be the appearance of a second “quantum scale”  $\tilde{\beta}$ . Moreover, the rigid limit will be independent only in the way  $\mathbb{P}^1$  is fibered over  $F_n$ , but it will not be independent of  $n$ . One can easily see that the resulting rigid surfaces have the form

$$W_{K3} = z + w + \frac{\beta}{z} + \frac{z^n \tilde{\beta}}{w} + (x^2 + u) = 0 ,$$

which give quartic  $K3$ ’s for  $n = 0, \dots, 3$ . Obviously one has universality only in the limit  $\tilde{\beta} \rightarrow 0$ , where one recovers the well-known SW gauge coupling, and indeed one may view these theories as coming from fibrations of the SW geometry over a further  $\mathbb{P}^1$  (whose size is governed by  $\tilde{\beta}$ ). Accordingly each of the two monopole singularities is split into two further singularities, the scales of the two independent splittings being  $\beta$  and  $\tilde{\beta}$ .

For higher rank groups (ie., general ADE singularities fibered over  $\mathbb{P}^2$ ), we will generically not find  $K3$  surfaces. This is similar to  $d = 4$ , where the SW curves for  $SU(n)$  are not Calabi-Yau (ie.,  $c_1 \neq 0$ ) for  $n > 2$ . For example, fibering an  $A_2$  singularity we will end up with  $W = z + w + \frac{\beta^6}{zw} + (x^3 + ux + v) = 0$ , and the homogenous form of this surface in  $W\mathbb{P}(1, 1, 1, 1; 6)$  does not give a  $K3$ . We expect it to have not one, but two holomorphic  $(2, 0)$  forms (and analogously  $n - 1$  holomorphic  $(2, 0)$  forms for  $A_{n-1}$  singularities, so that  $\partial_{u_{k+1}} \lambda_2 = \Omega_{2,0}^{(k)}$ ,  $k = 1, \dots, n - 1$ ).

For arbitrary groups  $G$ , there will be vectors of 4-fluxes and potentials  $\widetilde{W}_i \sim (\vec{\alpha}_i \cdot \vec{\Sigma}) \log(\vec{\alpha}_i \cdot \vec{\Sigma})$ , of which  $\text{rank}(G)$  are independent. Integrating the topological term  $\frac{\theta}{2\pi} (\vec{F} \cdot \vec{\nu})$  over 2d space time, we see that the 4-fluxes  $\vec{\nu}$  must indeed be quantized and dual to the instanton numbers  $\vec{n} = \frac{1}{2\pi} \int d^2 x \vec{F}$ . They thus naturally lie on the corresponding weight lattice,  $\vec{\nu} \in \Lambda_w^G$ , and we expect this also to be reflected by the intersection properties of the middle homology. Weyl invariance (up to theta-shifts) can be restored by simultaneously Weyl transforming  $\vec{\sigma}$  and the symmetry breaking fluxes  $\vec{\nu}$ , and in this way the effective theory exhibits the underlying group  $G$ .

## 6. Conclusions

We have been investigating a field theory limit of type IIA string compactification on certain 4-folds that have as compact piece a fibration of  $\mathbb{P}^1$  over  $\mathbb{P}^2$ . The effective action is given by a  $U(1) \times U(1)$  gauge theory, where one  $U(1)$  factor decouples. It is characterized by a complex Fayet-Iliopoulos coupling  $\tau(\sigma)$  given in (4.15), which displays a logarithmic one-loop piece plus further non-perturbative corrections. The twisted chiral superpotential  $\widetilde{W}(\Sigma)$  is given by the middle period of an auxiliary  $K3$  surface. This surface encodes non-perturbative information about the Coulomb branch of the effective  $U(1)$  theory, and thus is the two-dimensional analog of the SW curve in four dimensions.

Although *a priori* the geometrical setup of this kind of theories appears to be similar to that of  $N = 2$  gauge theories in  $d = 4$  [10], the theories turn out to be remarkably different. What we find in  $d = 2$  are not just the usual  $SU(2)$  (or more generally ADE) gauge theories or Grassmannian models, but theories whose structure is richer.

An important difference as compared to 4d is that the perturbative piece of such a 2d field theory does not fully determine the non-perturbative theory. That is, the logarithmic one-loop term arises from wrappings of world-sheet instantons around the fiber, and is universal for a given fibered ADE singularity. On the other hand, wrappings around the various classes of the base manifold give non-perturbative corrections, and different choices for the base provide different “non-perturbative completions” of the perturbative theory. This is extra information which goes beyond the perturbative definition of the theory, but which is required for the global consistency of the full theory. It may simply be that one cannot well define the 2d theory unless it is embedded in a larger consistent framework, and, unlike as in  $d = 4$  where there is only one choice for the base, this embedding turns out to be ambiguous.

Besides continuous moduli, the effective theory also possess discrete moduli to play with, the 4-fluxes  $\nu$ .

If there is no 4-flux turned on, the 2d theory is essentially a reduction of an ADE gauge theory in six dimensions on some 2-fold base. For such a theory the holomorphic potential  $\widetilde{W}(\Sigma)$  that we have been computing does not appear in the effective lagrangian. Still, even for vanishing  $\nu$ , there remains some structure in the theory, namely a non-trivial Kähler potential and a non-trivial complex structure moduli space. The Kähler potential is not protected from corrections, and it is thus

not clear what conclusions can be drawn from the geometric expression for it given in (4.16).

In our example, the complex structure quantum moduli space has the form of a non-abelian orbifold,  $\mathcal{M} = \mathbb{H}/\Gamma_1$ , where  $\Gamma_1$  is the monodromy group of our problem. Strictly speaking, since there is no good notion of a VEV in two dimensions, the moduli space should rather be viewed as a target space of a sigma-model, to be functionally integrated over. Remember however that a fivebrane instanton induced potential cannot be excluded if  $\nu = 0$  (see section 3), and is in fact expected to appear [29,30]. Such a potential would substantially change the vacuum structure. It is not clear to us, though, how to compute it using mirror symmetry.

More interestingly, the theory looks very different if we turn on  $\nu$ : then a non-trivial potential  $\widetilde{W}(\Sigma)$  is generated, which breaks the ADE symmetry and which reflects extra matter fields that were not present before. At the perturbative level the theory is simply a  $U(1)$  gauge theory with some charged matter, and hence similar to a  $CP^n$  model. Non-perturbatively, a non-vanishing 4-flux leads to a spontaneous breakdown of supersymmetry, unless a bare coupling is added by hand. In this case a mass gap appears and supersymmetric ground states are naturally associated with singularities in the moduli space. At large distances the Kähler potential becomes then irrelevant, and the theory becomes a topological field theory with Chern-Simons lagrangian  $\mathcal{L} = \nu \widetilde{W}(\Sigma)$ . It can probably be interpreted along the lines of [22] as some kind of abelian WZW model at “level  $\nu$ ”.

We see that switching on the 4-flux is a quite drastic operation, and resembles a bit to switching on a shift vector in orbifold compactifications. It would seem worthwhile to investigate the rôle of the 4-flux also from that perspective.

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